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APPENDIX C

Explicit Finite-Volume Time-Marching Calculations of Total Temperature Distributions in Turbulent Flow

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1. SUMMARY

A method has been developed which calculates two-dimensional, transonic, viscous flow in ducts. The finite volume, time marching formulation is used to obtain steady flow solutions of the Reynolds-averaged form of the Navier Stokes equations. The entire calculation is performed in the physical domain. This paper investigates the introduction of a new formulation of the energy equation which gives improved transient behavior as the calculation converges. The effect of variable Prandtl number on the total temperature distribution through the boundary layer is also investigated.

A turbulent boundary layer in an adverse pressure gradient ($M = 0.55$) is used to demonstrate the improved transient temperature distribution obtained when the new formulation of the energy equation is used. A flat plate turbulent boundary layer with a supersonic freestream Mach number of 2.8 is used to investigate the effect of Prandtl number on the distribution of properties through the boundary layer. The computed total temperature distribution and recovery factor agree well with the measurements when a variable Prandtl number is used through the boundary layer.

2. INTRODUCTION

This paper is an extension of the work reported elsewhere in this conference [1]. A review of the features of the new method will be included here but a more complete discussion may be found in references 1 and 2.

The features of the current method can be summarized as follows. Control volumes are chosen so that smoothing of flow properties, typically required for stability, is not needed. Different time steps are used in the different governing equations to improve the convergence speed of the viscous calculations. A multi-volume method for pressure changes in the boundary layer allows calculations which use very long and thin control volumes (length/height = 1000).

3. GOVERNING EQUATIONS

The unsteady forms of the continuity equation, the x-momentum equation, the y-momentum equation, and the energy equation, in integral form, are used to obtain steady-state solutions for flow through 2-dimensional ducts. This approach differs from our previous work [1] where the assumption of constant total temperature was used instead of the full energy equation. The ideal gas equation of state and a Prandtl mixing length turbulence model [1] complete the governing equations needed to solve for the unknown variables ρ, u, v, P, μ, and T.

For a finite control volume where we can assign one value of density to the control volume, and for a finite time step, δt, continuity states that,

$$\rho^{n+1} - \rho^{n} = \delta\rho = -\left[\iint \rho \, \underline{u} \cdot d\,\underline{A} \right] \frac{\delta t}{\delta Vol} \qquad (1)$$

where the integral is evaluated explicitly at the current time step, n. In arriving at an expression which relates the pressure change directly to the continuity error, we will assume that changes in temperature are small in comparison to other changes for one time step. Thus, we can relate changes in pressure to changes in density through the ideal gas equation of state.

$$P^{n+1} - P^{n} = \delta P = -RT\left[\iint \rho \, \underline{u} \cdot d\,\underline{A} \right] \frac{\delta t}{\delta Vol} \qquad (2)$$

For the method introduced in the current work, a non-conservative form of the unsteady momentum equation is used. The non-conservative form is used because it allows the current method to use different time steps for the continuity, momentum, and energy equations. The difference between the non-conservative and conservative forms of the unsteady momentum

equation is associated with the unsteady and convective terms. Specifically, we note that

$$\frac{\partial(\rho \underline{u})}{\partial t} + \nabla \cdot \rho \, \underline{u} \, \underline{u} = \rho \frac{\partial \underline{u}}{\partial t} + \rho \, \underline{u} \cdot \nabla \underline{u} \tag{3}$$

and the right hand side of Eq. 3 can be rewritten as

$$\rho \frac{\partial \underline{u}}{\partial t} + \rho \, \underline{u} \cdot \nabla \underline{u} = \rho \frac{\partial \underline{u}}{\partial t} + \nabla \cdot \rho \, \underline{u} \, \underline{u} - \underline{u} \, (\nabla \cdot \rho \, \underline{u}) \tag{4}$$

When the right hand side of Eq. 4 is combined with the pressure and viscous terms, the momentum equation in integral form becomes

$$(\underline{u})^{n+1} - (\underline{u})^{n} = \delta(\underline{u}) = \Big[-\iint \rho \, \underline{u} \, \underline{u} \cdot d \, \underline{A} + \overline{u} \iint \rho \, \underline{u} \cdot d \, \underline{A} - \iint P\delta_{ij} \cdot d \, \underline{A} + \iint (\mu \nabla \underline{u} + \mu \, \overline{\nabla \underline{u}^{T}}) \cdot d \, \underline{A} \Big] \frac{\delta t}{\delta Vol} \tag{5}$$

To maintain stability, the properties must be updated in the proper sequence. In the current method, the sequence is:

1. update the pressure from the continuity equation;
2. update the velocities from the momentum equations using the new pressure and old velocities and old density;
3. update the density from the ideal gas equation of state;
4. update the temperature from the energy equation.

4. ENERGY EQUATION

For many calculations of transonic viscous flow, the assumption of constant total temperature will give a sufficient representation of the energy equation in the flow field. By assuming constant total temperature, the computations are less expensive to run and the computer storage requirements are less. The assumption of constant total temperature is usually satisfactory if:

1. an adiabatic wall is assumed in the calculations;
2. no work is done on the fluid at the solid boundaries;
3. the Mach numbers in the flow fields are low enough that total temperature gradients within the boundary layer are small;
4. the Prandtl number is approximately 1.0.

For a Prandtl number of 0.9, the solution should not deviate greatly from the constant total temperature assump-

tion. However for high speed flow, the energy equation should be included in the calculations especially if the Prandtl number deviates greatly from 1.

Two forms of the integral formulation of the energy equation will be derived next.

The energy equation in differential form is

$$\frac{\partial E_t}{\partial t} + \nabla \cdot E_t \underline{u} = -\nabla \cdot \underline{q} + \nabla \cdot \left[\underline{u} \cdot (\mu \nabla \underline{u} + \mu \overline{\nabla \underline{u}^T}\right] - \nabla \cdot P \underline{u} \tag{6}$$

where the total energy per unit volume, E_t, is

$$E_t = \rho \left(e + \tfrac{1}{2}(u^2 + v^2)\right) = \rho e_t \tag{7}$$

The left hand side of Eq. 6 can be rewritten as

$$\frac{\partial E_t}{\partial t} + \nabla \cdot E_t \underline{u} = \frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot \rho e_t \underline{u} \tag{8}$$

and

$$\frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t) \underline{u} = \rho \frac{\partial e_t}{\partial t} + \rho \underline{u} \cdot \nabla e_t \tag{9}$$

then, expanding the right hand side of Eq. 9, we get,

$$\rho \frac{\partial e_t}{\partial t} + \rho \underline{u} \cdot \nabla e_t = \rho \frac{\partial e_t}{\partial t} + \nabla \underline{u} e_t - e_t(\nabla \cdot \rho \underline{u}) \tag{10}$$

The procedure just outlined is identical to what was done to the unsteady and convective terms in the momentum equation (see Eqs. 3,4).

The heat flux vector, $\underline{q}$, can be represented as

$$\underline{q} = -k\nabla T \tag{11}$$

Substituting Eqs. 8-11 into Eq. 6, we get

$$\begin{aligned} \rho \frac{\partial e_t}{\partial t} &= -\nabla \cdot \rho \underline{u} e_t + e_t(\nabla \cdot \rho \underline{u}) - \nabla \cdot (-k\nabla T) \\ &+ \nabla \cdot \left[\underline{u} \cdot \left(\mu\nabla \underline{u} + \mu \overline{\nabla \underline{u}^T} \right)\right] - \nabla \cdot P \underline{u} \end{aligned} \tag{12}$$

The integral form of the energy equation is then

$$\rho \frac{\delta e_t}{\delta t} \times \delta Vol =$$

$$- \iint \rho \underline{u} e_t \cdot d \underline{A} + \bar{e}_t \iint \rho \underline{u} \cdot d \underline{A} - \iint -k\nabla T \cdot d \underline{A} \tag{13}$$

$$+ \iint \left[\underline{u} \cdot \left(\mu \nabla \underline{u} + \mu \overline{\nabla \underline{u}^T} \right) \cdot d\,\underline{A} - \iint P\,\underline{u} \cdot d\,\underline{A} \right.$$

where $\bar{e}_t$, is an average value for the control volume. As with the momentum equation, Eq. 13 has a term $\bar{e}_t \iint \rho\,\underline{u} \cdot d\,\underline{A}$, which removes the continuity error contribution to the energy error.

This form of the energy equation, when incorporated into the current method, behaved poorly. Initially there were large errors in continuity and momentum and these large errors acted through this energy equation to cause errors in the total energy for a control volume. This interaction was destabilizing.

An alternative form of the energy equation will now be derived. This alternative form has enhanced convergence properties when compared with the above formulation. Briefly, the energy equation is reformulated so that changes in total enthalpy, h_t, are calculated rather than changes in total energy, e_t, which was done previously. This allows us to see the terms which cause departures from uniform total temperature - for both the steady state solution and the transient solution.

The total enthalpy can be defined in terms of the total energy and the static temperature

$$h_t = e_t + P/\rho \tag{14}$$

or

$$h_t = e_t + RT \tag{15}$$

Taking the derivative with respect to time and multiplying by the density, we get

$$\rho \frac{\partial h_t}{\partial t} = \rho \frac{\partial e_t}{\partial t} + \rho\, R \frac{\partial T}{\partial t} \tag{16}$$

The static temperature T can be represented in terms of the total enthalpy and the absolute velocity as

$$T = \frac{h_t}{C_p} - \frac{V^2}{2C_p} \tag{17}$$

Therefore

$$\frac{\partial T}{\partial t} = \frac{1}{C_p} \frac{\partial h_t}{\partial t} - \frac{V}{C_p} \frac{\partial V}{\partial t} \tag{18}$$

Substituting Eq. 18 into Eq. 16, we obtain

$$\rho \frac{\partial e_t}{\partial t} = \frac{\rho}{\gamma} \frac{\partial h_t}{\partial t} + \rho \frac{R}{C_p} V \frac{\partial V}{\partial t} \tag{19}$$

where γ is the ratio of specific heat capacities and V is the magnitude of the velocity vector. Using equations (19) and (14) to eliminate e_t from equation (12) we get

$$\frac{\rho}{\gamma}\frac{\partial h_t}{\partial t} = -\nabla\cdot\rho\,\underline{u}\,h_t + (h_t - \frac{P}{\rho})(\nabla\cdot\rho\,\underline{u}) + \nabla\cdot k\nabla T$$

$$+ \nabla\cdot[\underline{u}\cdot(\mu\nabla u + \mu\overline{\nabla\,\underline{u}^T}] - \rho\,\frac{R}{C_p}\,V\,\frac{\partial V}{\partial t} \tag{20}$$

Using $h_t = C_p\,T + V^2/2$ and $k = \mu\,C_p/Pr$, $k\nabla T$ may be replaced by

$$k\nabla T = \frac{\mu}{Pr}\,\nabla h_t - \frac{\mu}{Pr}\,\nabla(\frac{V^2}{2}) \tag{21}$$

and from continuity we may replace $\nabla\cdot\rho\,\underline{u}$ with $-\partial\rho/\partial t$.

Therefore the energy equation written as a conservation equation for total enthalpy is

$$\frac{\rho}{\gamma}\frac{\partial h_t}{\partial t} = \underset{(I)}{-\nabla\cdot\rho\,\underline{u}\,h_t} + \underset{(II)}{h_t(\nabla\cdot\rho\,\underline{u})} + \underset{(III)}{\nabla\cdot\frac{\mu}{Pr}\,\nabla h_t}$$

$$+ \underset{(IV)}{\nabla\cdot\mu(1 - \frac{1}{Pr})\nabla(\frac{V^2}{2})} + \underset{(V)}{\nabla\cdot\mu(\underline{u}\cdot\nabla)\,\underline{u}} + \underset{(VI)}{\frac{P}{\rho}\frac{\partial\rho}{\partial t}} - \underset{(VII)}{\frac{\rho R}{C_p}\,V\,\frac{\partial V}{\partial t}} \tag{22}$$

Terms I and II when combined give $-\rho\,\underline{u}\cdot\nabla h_t$. Therefore terms I + II and III contain h_t only in the form ∇h_t. Thus, when these are the only important terms in the equation, flow with uniform total temperature at the inlet will retain this uniform total temperature provided that the boundary conditions are consistent with this.

Term IV is a viscous transport term for total enthalpy when the Prandtl number is other than 1. Term V is another viscous transport term. It however contains the expression $(\underline{u}\cdot\nabla)\,\underline{u}$ which is the gradient of the velocity in the direction of the velocity; these gradients are usually small compared with other velocity gradients. Since terms IV and V have the form $\nabla\cdot(\ )$, they are not source terms, rather they can only redistribute the total enthalpy. Terms VI and VII on the other hand have the form of source terms. Relative to the steady state, they are proportional to the continuity error and the momentum error, respectively. We may write them as

$$M = \ell\,\frac{\partial\rho}{\partial t} + m\,\frac{\partial V}{\partial t} \tag{23}$$

At the steady state, Eq. 22 becomes

$$0 = -\nabla\cdot\rho\,\underline{u}\,h_t + \nabla\cdot\frac{\mu}{Pr}\nabla h_t + \nabla\cdot\mu\left(1-\frac{1}{Pr}\right)\nabla\left(\frac{V^2}{2}\right) + \nabla\cdot\mu(\underline{u}\cdot\nabla)\,\underline{u} \tag{24}$$

Therefore we may artibrarily alter the variables l and m in Eq. 23 and the steady form of the energy equation, Eq. 24, will be obtained for converged solutions. The transient behavior of h_t is improved in the calculation procedure by choosing $l = m = 0$, i.e. by omitting the transient source terms in the enthalpy equation.

In integral form then the equation for enthalpy changes is

$$\rho\,\frac{\delta h_t}{\delta t}\,\delta Vol = \gamma\Big\{-\iint \rho\,\underline{u}\,h_t\cdot d\,\underline{A} + \bar{h}_t \iint \rho\,\underline{u}\cdot d\,\underline{A} \tag{25}$$

$$+ \iint \frac{\mu}{Pr}\nabla h_t\cdot d\,\underline{A} + \iint \left[\left(\mu-\frac{\mu}{Pr}\right)\underline{u}\cdot\overline{\nabla\,\underline{u}^T} + \mu\,\underline{u}\cdot\nabla\,\underline{u}\right]\cdot d\,\underline{A}$$

where $\mu = \mu_\ell + \mu_t$ and $\frac{\mu}{Pr} = \frac{\mu_\ell}{Pr_\ell} + \frac{\mu_t}{Pr_t}$.

The time step used for the enthalpy equation is the same as for the momentum equation. If the transient source term $\frac{P}{\rho}\frac{\partial\rho}{\partial t}$ had been retained in the enthalpy equation, it would have been necessary to link the continuity and energy equation time steps. Omitting this term allows us to use different time steps for the energy equation.

5. TEST CASES

Two test cases will be used to explore various aspects of the more complete form of the energy equation, Eq. 25, discussed previously.

5.1 Turbulent Boundary Layer in an Adverse Pressure Gradient

The geometry and grid used in this test case are shown in Fig. 1. Flow in this geometry was used in Ref. 1 to test the accuracy of the new computational scheme. In Ref. 1 the velocities in the duct were low enough that the flow could be treated as incompressible. Here, the inlet freestream Mach number was increased to 0.55. The purpose of this test case was to illustrate the advantage of the new formulation of the energy equation.

The static temperatures presented in Fig. 2 are from calculations after 500 iterations. It can be clearly seen that the new formulation, Eq. 25, gives a better transient

solution to the energy equation and it should result in a reduction in the computer time required to reach a steady state solution. Fig. 3 shows the corresponding total temperature profiles for the two formulations of the energy equation.

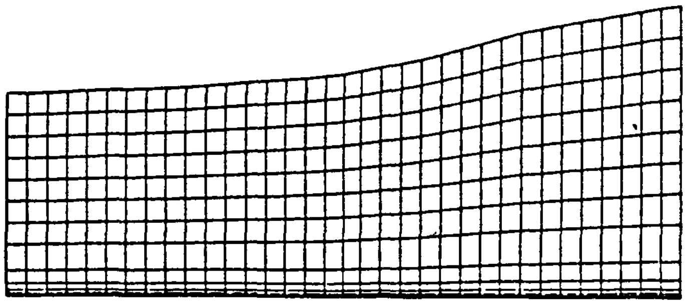

Fig. 1 Grid and Geometry Used to Demonstrate the Advantages of the New Formulation of the Energy Equation

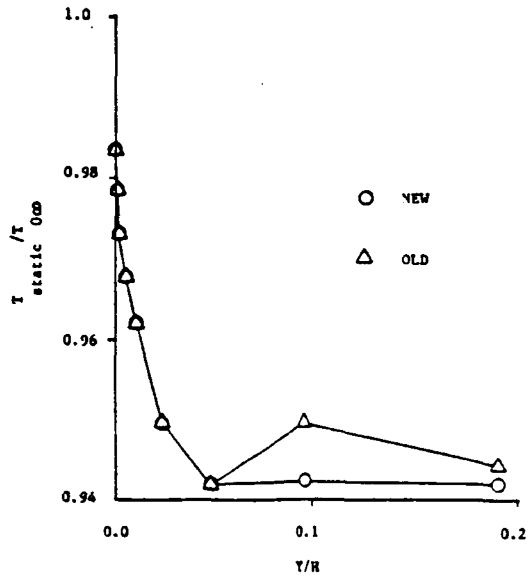

Fig. 2 Static Temperature Distribution Through the Boundary Layer at M=0.55, x=200 mm, after 500 iterations

5.2 Flat Plate Turbulent Boundary Layer at M = 2.8

Van Driest [3] presents the total temperature distribution within a flat plate turbulent boundary layer with a freestream Mach number of 2.8. The experimental total temperature distribution is shown in Fig. 4. The geometry and grid for these calculations are shown in Fig. 5. The height of the duct was 63.5 mm and the length of the duct was 254 mm. The computational grid shown in Fig. 5 consists of 21 axial grid points and 14 transverse grid points. The inlet boundary layer thickness of 6.35 mm was 10% of the duct

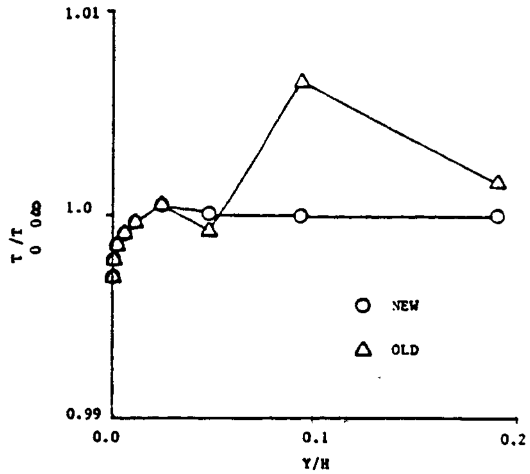

Fig. 3 Total Temperature Distribution Through the Boundary Layer at M=0.55, x=200 mm, after 500 iterations

height. The Reynolds number based upon axial distance is approximately 10^7. To stabilize these supersonic flow calculations, the upwind effective density method was used [2]. This means that an effective density used at a grid point is calculated with the ideal gas equation of state using the pressure from the next upstream grid point. The inlet velocity, total temperature, and total pressure were specified at the upstream boundary. Three calculations were performed with different assumptions about the turbulent Prandtl number. These assumptions were

1. $Pr_t = 0.90$ $\quad Pr_\ell = 0.73$
2. $Pr_t = 0.73$ $\quad Pr_\ell = 0.73$
3. Pr_t varies linearly through the boundary layer from 0.9 at the wall to 0.66 in the freestream.

The turbulent Prandtl number is typically set equal to a constant of 0.9 in calculations [4]. The calculated total temperature distribution through this boundary layer using a constant turbulent Prandtl number of 0.9 is shown in Fig. 6 (represented as ▢). The recovery factor is calculated to be 0.920 which compares with the empirically determined value of 0.90. However, the distribution of total temperature through the boundary layer does not compare well with the experiment. If the turbulent Prandtl number is set equal to the laminar Prandtl number of 0.73, the total temperature distribution changes as seen in Fig. 6 (represented at ○'s). The distribution through the outer part of the boundary layer has improved but the recovery factor of 0.813 does not compare well with the experimental value of 0.90. Schlichting [5] notes that the turbulent Prandtl number is not constant through the boundary layer. The experiments of H. Ludwieg [6] for turbulent flow through a pipe show that the Prandtl number varies from approximately 0.9 at the pipe

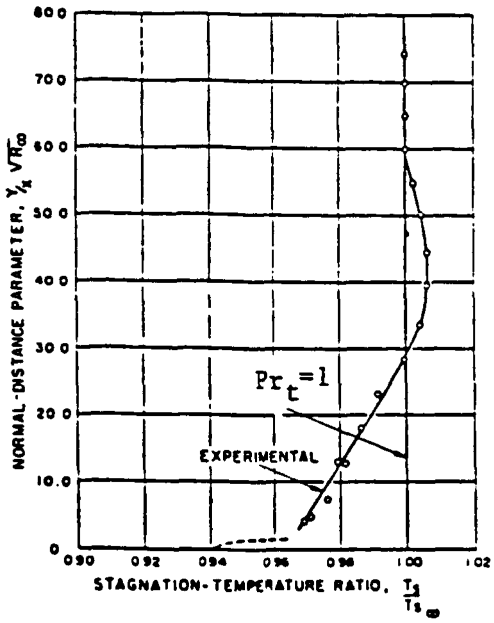

Fig. 4 Experimental Total Temperature Distribution in a Flat Plate Turbulent Boundary Layer $M=2.8$, $Re_x = 10^7$ after van Driest

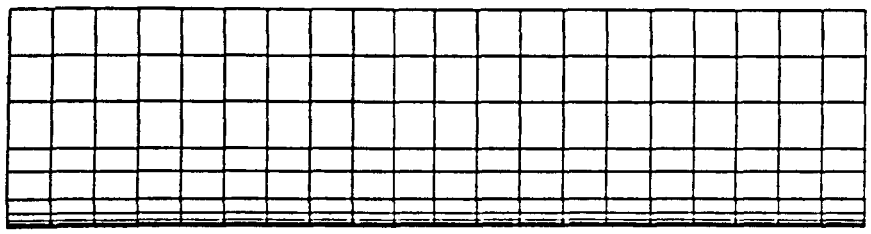

Fig. 5 Geometry and Grid For Boundary Layer Calculations at M=2.8

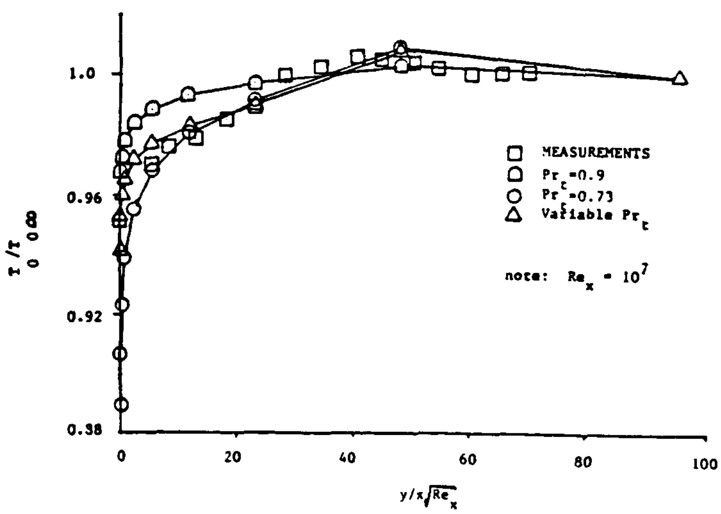

Fig. 6 Total Temperature Distribution For Flat Plate Boundary Layer at M=2.8

wall to 0.66 at the center of the pipe. This distribution is shown in Fig. 7. The variation is almost linear. For the third set of calculations, the Prandtl number was assumed to vary linearly through the boundary layer from 0.9 at the wall to 0.66 at the edge of the boundary layer. The total temperature distribution for this case is shown in Fig. 6 (represented as △'s). The total temperature distribution calculated using a variable Prandtl number is also compared with the experimental results in Fig. 8. Both the distribution of total temperature through the boundary layer and the recovery factor of 0.90 are in good agreement with the experimentally measured values.

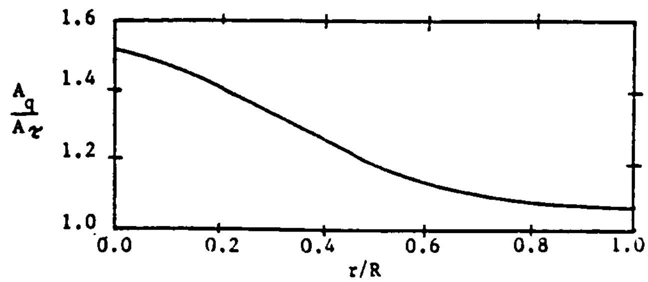

Fig. 7 Ratio of the Turbulent Transfer Coefficient Over the Length of a Radius in Turbulent Pipe Flow

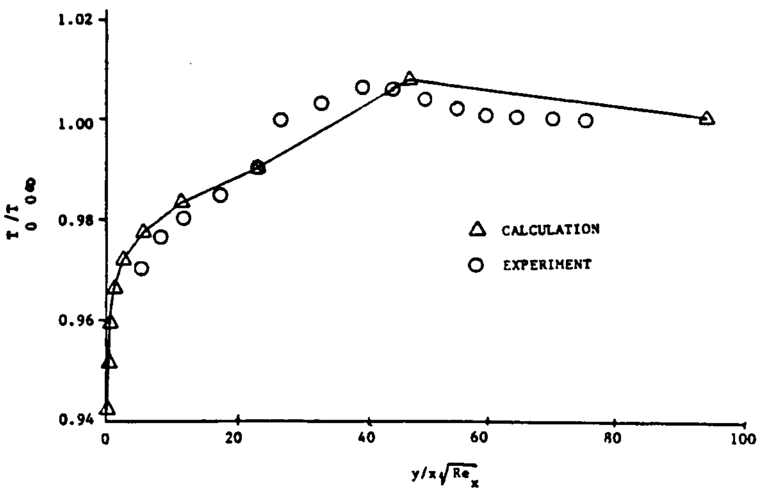

Fig. 8 Total Temperature Distribution For Flat Plate Boundary Layer at M=2.8 Computation vs. Experiment

6. CONCLUSIONS

A new formulation for the energy equation was introduced which has improved transient behavior when compared with the standard formulation. The new formulation removes the influences of continuity and momentum errors from the energy equation during transients in the solution.

For flat plate turbulent boundary layer flow with a freestream Mach number of 2.8, the calculated total temperature profile was improved by using a variable Prandtl number through boundary layer. The recovery factor of 0.90 agreed very well with the empirically determined value of 0.9.

7. ACKNOWLEDGEMENTS

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8. REFERENCES

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